Computing π with Forth

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Joel Shprentz

Strategy

- Quick review of techniques
- Avoid
  - Floating point
  - Very long integers
  - Trigonometry
  - Stochastic methods
- Favor
  - Simple algorithms
  - Sample code
  - Clear explantions

A Spigot Algorithm for the Digits of π

Stanley Rabinowitz and Stan Wagon

It is remarkable that the algorithm illustrated in Table 1, which uses no floating-point arithmetic, produces the digits of π. The algorithm starts with some 2s, in columns headed by the fractions shown. Each entry is multiplied by 10. Then, starting from the right, the entries are reduced modulo *den*, where the head of the column is *num*/*den*, producing a quotient q and remainder r. The remainder is left in place and $q \times num$ is carried one column left. This reduce-and-carry is continued all the way left. The tens digit of the leftmost result is the next digit of π. The process continues with the multiplication of the remainders by 10, the reductions modulo the denominators, and the augmented carrying.

TABLE 1. The workings of an algorithm that produces digits of π. The dashed line indicates the key step: starting from the right, entries are reduced modulo the denominator of the column head (25, 23, 21, ..., resp.), with the quotients, after multiplication by the numerator (12, 11, 10, ...), carried left. For example, the 20 in the $\frac{9}{19}$'s column yields a remainder of 1 and a left carry of $1 \cdot 9 = 9$. After the leftmost carries, the tens digits are 3, 1, 4, 1. To get more digits of π one must start with a longer string of 2s.

	Digits of π	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{4}{9}$	$\frac{5}{11}$	$\frac{6}{13}$	$\frac{7}{15}$	$\frac{8}{17}$	$\frac{9}{19}$	$\frac{10}{21}$	$\frac{11}{23}$	$\frac{12}{25}$
Initialize		2	2	2	2	2	2	2	2	2	2	2	2
$\times 10$		20	20	20	20	20	20	20	20	20	20	20	20
Carry	3	+10	+12	+12	+12	+10	+12	+7	+8	+9	+0	+0	+0
		30	32	32	32	30	32	27	28	29	20	20	20
Remainders		0	2	2	4	3	10	1	13	12	1	20	20
$\times 10$		0	20	20	40	30	100	10	130	120	10	200	200
Carry	1	+13	+20	+33	+40	+65	+48	+98	+88	+72	+150	+132	+96
		13	40	53	80	95	148	108	218	192	160	332	296
Remainders		3	1	3	3	5	5	4	8	5	8	17	20
$\times 10$		30	10	30	30	50	50	40	80	50	80	170	200
Carry	4	+11	+24	+30	+40	+40	+42	+63	+64	+90	+120	+88	+0

References

- Rabinowitz, Stanley and Wagon, Stan (1995) “A Spigot Algorithm for the Digits of π,” *The American Mathematical Monthly*, 102(3), pp. 195–203. doi: 10.2307/2975006.
- *Computing the Digits in π* by Carl D. Offner https://www.cs.umb.edu/~offner/files/pi.pdf
- *Computing Pi in C* by Ben Lynn https://crypto.stanford.edu/pbc/notes/pi/code.html
- *Pi Formulas* by Eric Weisstein https://mathworld.wolfram.com/PiFormulas.html
- *Calculation of the Digits of π by the Spigot Algorithm* by Alexander Bogomolny https://www.cut-the-knot.org/Curriculum/Algorithms/SpigotForPi.shtml

π in Base 10

- $\pi = 3.14159\ldots$
- $\pi = 3 + \frac{1}{10}(1 + \frac{1}{10}(4 + \frac{1}{10}(1 + \frac{1}{10}(5 + \frac{1}{10}(9 + \frac{1}{10}(\ldots))))))$

A Series for π

- $\frac{\pi}{2} = \sum_{i=0}^{\infty} \frac{i!}{(2i+1)!}$

- $\pi = 2 + \frac{1}{3}(2 + \frac{2}{5}(2 + \frac{3}{7}(2 + \frac{4}{9}(2 + \frac{5}{11}(2 + \frac{6}{13}(\ldots))))))$

Calculation of π is a base conversion

- From: (2, 2, 2, 2, 2, …) base $(\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{4}{9},\frac{5}{11},\frac{6}{13},\ldots)$
- To: (3, 1, 4, 1, 5, 9, …) base $\frac{1}{10}$

- Number of terms required depends on number of digits, n

$$\left\lfloor \frac{10n}{3} \right\rfloor + 1$$

Sample Run

```
pi@raspberrypi:~/forth/svfig-challenge-2 $ gforth pi.fs 1000
3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944
5923078164 0628620899 8628034825 3421170679 8214808651 3282306647 0938446095
5058223172 5359408128 4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091 4564856692 3460348610
4543266482 1339360726 0249141273 7245870066 0631558817 4881520920 9628292540
9171536436 7892590360 0113305305 4882046652 1384146951 9415116094 3305727036
5759591953 0921861173 8193261179 3105118548 0744623799 6274956735 1885752724
8912279381 8301194912 9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132 0005681271 4526356082
7785771342 7577896091 7363717872 1468440901 2249534301 4654958537 1050792279
6892589235 4201995611 2129021960 8640344181 5981362977 4771309960 5187072113
4999999837 2978049951 0597317328 1609631859 5024459455 3469083026 4252230825
3344685035 2619311881 7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778 1857780532 1712268066
1300192787 6611195909 216420198
```

Compare with Reference

Digits of Pi

First 100 decimal places

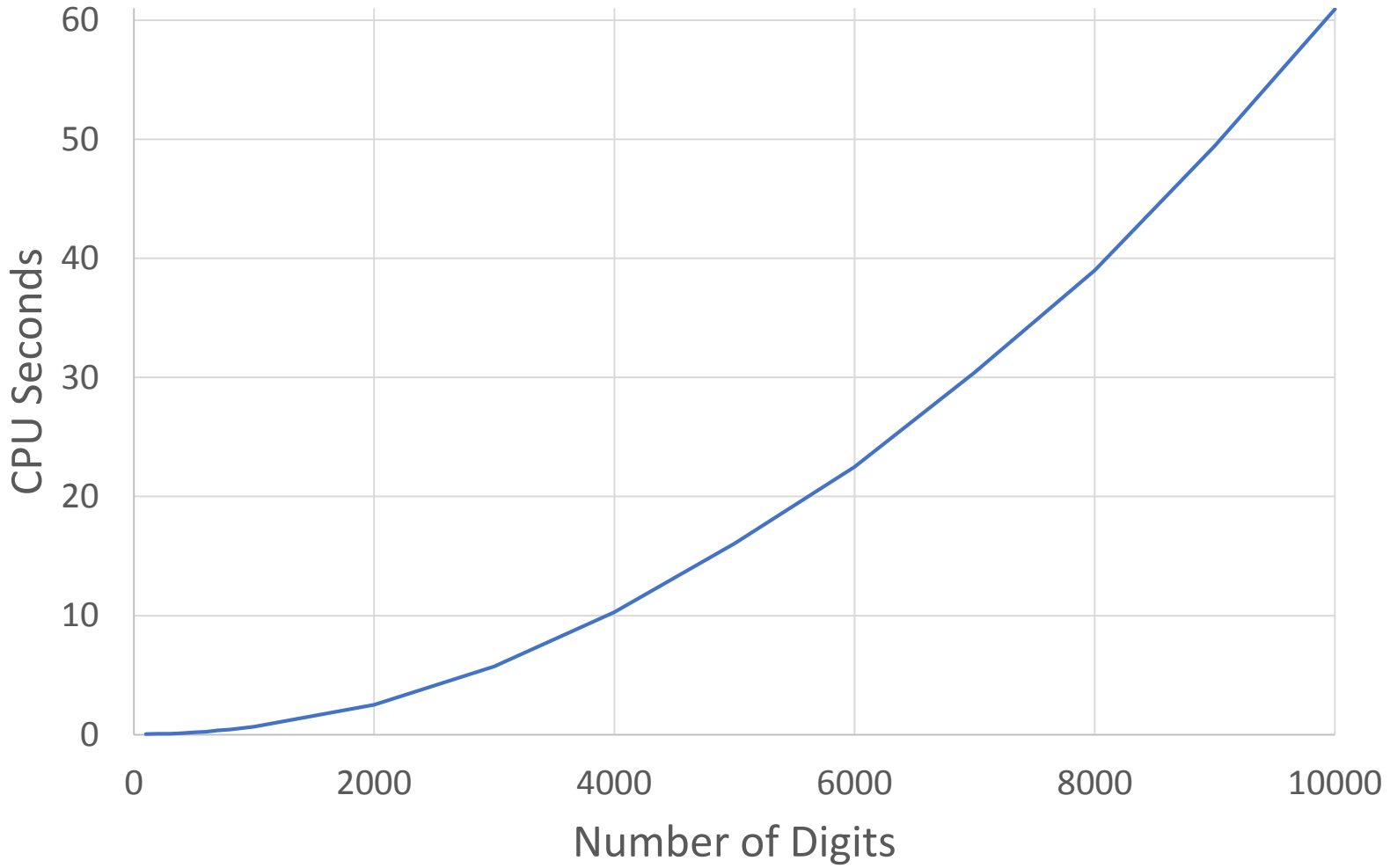
3.1415926535 8979323846 2643383279 5028841971 6939937510
5820974944 5923078164 0628620899 8628034825 3421170679 ...

First 1000 decimal places

3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944
5923078164 0628620899 8628034825 3421170679 8214808651 3282306647 0938446095
5058223172 5359408128 4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091 4564856692 3460348610
4543266482 1339360726 0249141273 7245870066 0631558817 4881520920 9628292540
9171536436 7892590360 0113305305 4882046652 1384146951 9415116094 3305727036
5759591953 0921861173 8193261179 3105118548 0744623799 6274956735 1885752724
8912279381 8301194912 9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132 0005681271 4526356082
7785771342 7577896091 7363717872 1468440901 2249534301 4654958537 1050792279
6892589235 4201995611 2129021960 8640344181 5981362977 4771309960 5187072113
4999999837 2978049951 0597317328 1609631859 5024459455 3469083026 4252230825
3344685035 2619311881 7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778 1857780532 1712268066
1300192787 6611195909 2164201989

http://www.math.com/tables/constants/pi.htm#d

Time Scales with Square of Number of Digits



Gforth on a
Raspberry Pi 4